Chapter Four - Compact Spaces

Definition. A collection $\mathcal{C}$ of subsets of a space X such that $X = \cup\mathcal{C}$ is called a **cover** of X. A cover of a space that consists of open sets is called an **open cover.**

Definition. A topological space is **compact** if every open cover of X includes a finite subcollection that is also a cover of X.

Proposition 4.1. With usual topology, every closed interval $[a,b]$ is compact.

Proposition 4.2. Suppose A is a subspace of a topological space X. Then the subspace A is compact if and only if it is true that for every collection $\mathcal{C}$ of open subsets of X with $A \subset \cup\mathcal{C}$, there is a finite subcollection $\mathcal{F} \subset \mathcal{C}$ such that $A \subset \cup\mathcal{F}$.

Examples 4.3

a)Every finite space is compact.

b)The real line with the usual topology is not compact, because $\mathcal{C} = \{(-r,r) : r \text{ real}\}$ is an open cover, and no finite subcollection of $\mathcal{C}$ is a cover.

Theorem 4.4. Suppose $f : X \to Y$ is a continuous function from one topological space into another. If A is a compact subspace of X, then $f(A)$ is compact.

Theorem 4.5. Every closed subset of a compact space is compact.

Example 4.6. Let X be an infinite set and let $\mathcal{T} = \{U \subset X\colon X\backslash U \text{ is finite.}\} \cup \{\emptyset\}$. Then $\mathcal{T}$ is a topology for X, and every subset of X is compact. [**Note.** This topology has a name; it is called the **cofinite** topology.]

Definition. A topological space X is a **Hausdorff** space if for every $x,y \in X$ with $x \neq y$, there are disjoint open sets U and V so that $x \in U$ and $y \in V$.

Theorem 4.7. Every compact subset of a Hausdorff space is closed.

Definition. A collection $\mathcal{K}$ of subsets of a set X is said to have the **finite intersection property** if the intersection of any finite subcollection of $\mathcal{K}$ is nonempty.

Theorem 4.8. A space X is compact if and only if every collection of closed subsets with the finite intersection property has nonempty intersection.

Example 4.9. The half open interval $[0,1)$ with the usual topology is not compact. The collection of closed sets $\mathcal{K} = \{[a,1) : 0 < a < 1\}$ has the finite intersection property, but $\cap\mathcal{K} = \emptyset$.

Definition. A pseudometric space (X,d) in which $d(x,y) = 0$ only if $x = y$ is called a **metric** space.

Proposition 4.10. A pseudometric space is a Hausdorff space if and only if it is a metric space.

Proposition 4.11. Every compact subset of a pseudometric space is bounded.

Proposition 4.12. A subset of the real numbers with the usual pseudometric is compact if and only if it is closed and bounded.