Chapter Five - Product Spaces

Definitions. Suppose $\{X_a : a \in A\}$ is a collection of sets. The set Z of all functions $f : A \to \cup\{X_a : a \in A\}$ such that $f(a) \in X_a$ for each $a \in A$ is called the **product** of the collection. We shall write $Z = \prod\{X_a : a \in A\}$.

For $a \in A$, the function $\pi_a : \prod\{X_a\} \to X_a$ defined by $\pi_a(f) = f(a)$ is the **projection** of the product $\prod\{X_a\}$ onto X_a.

Example 5.1. Let $A = \{1,2,3\}$ and let $X_1 = X_2 = X_3 =$ reals. Then each function f defined on A is simply a triple: $f = (f(1),f(2),f(3))$. The product $\prod\{X_a\}$ thus consists of all ordered triples of real numbers. The projection π_1 is given by $\pi_1(x,y,z) = x$.

Definition. If each X_a is a topological space, the weak topology for $\prod\{X_a\}$ by the collection of all projections is the **product topology.** In this case, the product set together with the product topology is called the **product space.**

Proposition 5.2. Let $Z = \prod\{X_a : a \in A\}$ and $b \in A$. Let $U_b \subset X_b$. Then

$$\pi_b^{-1}(U_b) = \prod\{Y_a : a \in A\},$$

where $Y_b = U_b$ and $Y_a = X_a$ for all $a \neq b$.

Proposition 5.3. The collection of all sets $\prod\{U_a : a \in A\}$, where U_a is an open subset of X_a and $U_a = X_a$ for all but a finite number of indices a, is a base for the product topology for $\prod\{X_a : a \in A\}$.

Theorem 5.4. Suppose a set X is endowed with the weak topology by a collection of functions $\{f_a : a \in A\}$, where $f_a : X \to Y_a$. If $f : Z \to X$ is a function from a topological space Z into X, then f is continuous if and only if each composition $f_a \circ f : Z \to Y_a$ is continuous.

Corollary 5.5. If $f : Z \to \prod\{X_a : a \in A\}$ is a function from a topological space Z into a product space, then f is continuous if and only if each "coordinate" function $\pi_a \circ f$ is.

Theorem 5.6. A product of a countable collection of pseudometric spaces is a pseudometric space.

Proposition 5.7. A product of Hausdorff spaces is Hausdorff.

Theorem 5.8. The usual metric topology for $\mathbf{R}^n$ is the product topology.

Theorem 5.9. The product of a collection of topological spaces is compact if and only if each space in the collection is compact. [This is the celebrated **Tychonoff Product**

Theorem.]

Theorem 5.10. A subset of $\mathbf{R}^n$ is compact if and only if it is closed, and bounded in the usual pseudometric.