Chapter Six - Sequences

Definition. A **sequence** in a set S is a function from the positive integers Z_+ into S. If $f : Z_+ \to S$ is a sequence, we usually write x_n for $f(n)$ and denote the sequence by (x_n).

Definitions. If $A \subset X$ and (x_n) is a sequence in X, then we say (x_n) is **eventually** in A if there is an integer N such that $x_n \in A$ for all $n \geq N$. We say that (x_n) is **frequently** in A if for each integer k, there is an integer $n \geq k$ such that $x_n \in A$.

Definitions. Suppose (x_n) is a sequence in a topological space X. A point $x \in X$ is a **limit** of (x_n) if (x_n) is eventually in every neighborhood of x, and we say the sequence **converges** to x. A point $x \in X$ is a **cluster point** of (x_n) if (x_n) is frequently in every neighborhood of x, and we say the sequence **clusters** at x.

Examples 6.1.
a)Let X be any set with the trivial topology. Then every point of X is a limit of every sequence.
b)Let X be any set with the discrete topology. Then a sequence converges to x only if it is eventually x.
c)Let X be the reals with the usual topology, and let (x_n) be the sequence defined by setting $x_n = (-1)^n + (1/n)$. Then 1 and -1 are cluster points, and no point is a limit.

Proposition 6.2. A point x in a pseudometric space (X,d) is a limit of the sequence (x_n) if and only if it is true that for each $\varepsilon > 0$, there is an integer N so that $d(x_n,x) < \varepsilon$ for all $n \geq N$.

Proposition 6.3. A sequence in a Hausdorff space has at most one limit.

Theorem 6.4. Suppose p is a cluster point of a sequence in a subset A of a topological space X. Then p is in the closure of A.

Theorem 6.5. In a pseudometric space, a point p is in the closure of a set A if and only if there is a sequence in A that converges to p.

Example 6.6. Let X be the reals with the cocountable topology

$$\boldsymbol{T} = \{U : X \backslash U \text{ is countable}\} \cup \{\emptyset\}.$$

Let $A = [0,1]$. The $x = 2$ is an element of clA, but there is no sequence in A that converges to 2.

Theorem 6.7. Suppose $f : X \to Y$ is a continuous function from one topological space into another. If (x_n) is a sequence in X that converges to x, the converges sequence $(f(x_n))$ converges to $f(x)$.

Theorem 6.8. Suppose $f : X \to Y$ is a function from a pseudometric space into a topological space. If for every convergent sequence (x_n) in X, the sequence $(f(x_n))$ converges to $f(x)$, where x is a limit of (x_n), then f is continuous.

Example 6.9. Let X be the reals with the cocountable topology, and let Y be the reals with the usual topology. Let $f : X \to Y$ be the identity function $f(x) = x$. The for every convergent sequence (x_n) in X, the sequence $(f(x_n))$ converges to $f(x)$, but f is not continuous.

Theorem 6.10. If X has the weak topology by a collection $\{f_a : a \in A\}$ of functions $f_a : X \to Y_a$, then a sequence (x_n) in X converges to x if and only if for every $a \in A$, the sequence $(f_a(x_n))$ converges to $f_a(x)$.

Corollary 6.11. The sequence (z_n) in a product space $Z = \prod\{X_a : a \in A\}$ converges to $w \in Z$ if and only if each sequence of projections $(\pi_a(z_n))$ converges to $\pi_a(w)$.

Theorem 6.12. Every sequence in a compact space has a cluster point.

Theorem 6.13. Let (X, d) be a pseudometric space in which every sequence has a cluster point, and let $\mathcal{C}$ be an open cover for X. Then there is a number $\delta > 0$ such that for each $x \in X$, the cell $C(x; \delta) \subset U$ for some $U \in \mathcal{C}$.

Note. Theorem 6.13 is the famous Lebesgue lemma, and the number δ is called a **Lebesgue number** for the cover $\mathcal{C}$.

Definition. A psudometric space is **totally bounded** if for every $\varepsilon > 0$ there is a finite cover of X consisting of cells of radius ε.

Proposition 6.14. A pseudometric space in which every sequence has a cluster point is totally bounded.

Proposition 6.15. Every compact pseudometric space is totally bounded.

Theorem 6.16. A pseudometric space is compact if and only if every sequence in the space has a cluster point.

Theorem 6.17. If pseudometric space (X, d) is totally bounded, then there is a countable base for the topology generated by d.