Chapter Seven - Complete Pseudometric Spaces

Definition. A sequence (x_n) in a pseudometric space (X,d) is a **Cauchy sequence** if for every $\varepsilon > 0$, there is an integer N so that $d(x_n,x_m) < \varepsilon$ for all $n,m \geq N$.

Proposition 7.1. Every convergent sequence in a pseudometric space is a Cauchy sequence.

Example 7.2. Let X be the irrational numbers with the usual pseudometric inherited from the space of real numbers. Then $(\sqrt{2}/n)$ is a Cauchy sequence that has no limit.

Definition. A pseudometric space in which every Cauchy sequence has a limit is a **complete pseudometric space**.

Proposition 7.3. Every sequence in a compact space has a cluster point.

Proposition 7.4. A Cauchy sequence with a cluster point converges.

Theorem 7.5. Every compact pseudometric space is complete.

Theorem 7.6. A closed subspace of a complete pseudometric space is complete.

Theorem 7.7. A complete subspace of a metric space is closed.

Example 7.8. Let X be the plane with the pseudometric $d(x,y) = |x_1 - y_1|$, where $x = (x_1,x_2)$ and $y = (y_1,y_2)$. Then the set $S = \{(t,0) : 0 \leq t \leq 1\}$ is a complete subspace of (X,d), but $clS = \{(x_1,x_2) : 0 \leq x_1 \leq 1, \text{ and } x_2 \text{ is real}\} \neq S$.

Definition. A function $f : (X,d) \to (Y,\rho)$ from one pseudometric space into another is a **contraction map** if there is a real number $k < 1$ such that $\rho(f(x),f(y)) \leq kd(x,y)$ for all $x,y \in X$.

Proposition 7.9. Let $f : X \to X$ be a contraction from a pseudometric space into itself. Let $x_0 \in X$ and for each positive integer n, let $x_n = f(x_{n-1})$. Then the sequence (x_n) is a Cauchy sequence.

Proposition 7.10. Let $f : (X,d) \to (X,d)$ be a contraction map from a complete pseudometric space into itself. Let $x_0 \in X$, and for each positive integer n, let $x_n = f(x_{n-1})$. The the sequence (x_n) converges, and for any limit z of the sequence, $d(z,f(z)) = 0$.

Theorem 7.11. Let $f : (X,d) \to (X,d)$ be a contraction map from a complete metric space into itself. Then there is exactly one point $z \in X$ such that $f(z) = z$. Moreover, if x_0 is any point in X and for each positive integer n, we define $x_n = f(x_{n-1})$, then the sequence (x_n) converges to z.

Note. This is the celebrated **Banach Fixed Point Theorem**.

Theorem 7.12. Let X be a complete pseudometric space, and let $\boldsymbol{D} = \{D_n : n \in Z_+\}$ be a countable collection of open dense subsets of X. Then $\cap\boldsymbol{D}$ is a dense subset of X.

Proposition 7.13. A subset A of a topological space X is open and dense if and only if $X \backslash A$ is closed and has empty interior.

Theorem 7.14. Let X be a complete pseudometric space, and let $\boldsymbol{F} = \{F_n : n \in Z_+\}$ be a countable collection of closed sets each of which has empty interior. Then $\cup\boldsymbol{F}$ has empty interior.

Note. Theorems 7.12 and 7.14 are versions of the famous **Baire Category Theorem**.