Chapter Eight - Linear Spaces

Definition. A **linear space** X is a set for which are defined an addition + making X an abelian group, and multiplication by scalars, satisfying the distributive laws $t(x+y) = tx+ty$ and $(s+t)x = sx+tx$, where t and s are scalars, $x,y \in X$, and satisfying $(st)x = s(tx)$ and $1x = x$.

Note. "Scalars" means complex numbers unless otherwise indicated.

Definition. A **linear subspace** of a linear space is a subset which with the same operations and scalars is a linear space.

Definition. A function $f : X \to Y$ from one linear space into another is a **linear function** if $f(x+y) = f(x)+f(y)$ and $f(tx) = tf(x)$ for all scalars t and all $x,y \in X$.

Definition. A set $M \subset X$, a linear space, is a **linear variety** if

$$M = x_0 + M_0 = \{x_0 + y : y \in M_0\}$$

for some $x_0 \in X$ and some linear subspace M_0.

Definition. A set $C \subset X$ is **convex** if $tx + (1-t)y \in C$ for all $x,y \in C$ and $0 \leq t \leq 1$.

Proposition 8.1. A linear variety is convex.

Proposition 8.2. Suppose X is a linear space, $a \in X$, $a \neq 0$, and $M = \{ta : \text{all } t\}$. Then M is a linear subspace. [M is traditionally called a **straight line through 0.**]

Definition. Linear subspaces M and N of a linear space X are said to be **complementary** if $M \cap N = \{0\}$ and $X = M+N$.

Note. If A and B are subsets of a linear space, $A+B = \{a+b : a \in A \text{ and } b \in B\}$.

Theorem 8.3. Subspaces M and N of a linear space X are complementary if and only if each $x \in X$ can be expressed uniquely as $x = m+n$, where $m \in M$ and $n \in N$.

Definition. A linear subspace complementary to a straight line through 0 is known as a **hyperplane through 0.**

Definitions. A linear variety $M = x_0 + M_0$ is a **straight line** if M_0 is a straight line through 0, and is a **hyperplane** if M_0 is a hyperplane through 0.

Theorem 8.4. A linear subspace M_0 of a space X is a hyperplane through 0 if and only if there is a nonconstant linear function $f : X \to S$ from X into the scalars such that $M_0 = f^{-1}(0)$.

Note. A linear function from a linear space into the scalars is frequently called a **linear functional.**

Corollary 8.5. A linear variety $M \subset X$ is a hyperplane if and only if there is a nonconstant linear function $f : X \to S$ from X into the scalars such that $M = f^{-1}(t)$ for some scalar t.

Theorem 8.6. Suppose N_0 is a linear subspace of the linear space X, and M_0 is a hyperplane through 0. If $M_0 \subset N_0$, then either $M_0 = N_0$ or $N_0 = X$.

Definition. Suppose X is a linear space that is also a topological space, and suppose the scalars are endowed with the usual topology. Then X is a **linear topological space** if the functions $F : X \times X \to X$, and $G : S \times X \to X$ given by $F(x,y) = x + y$ and $G(t,x) = tx$ are both continuous.

Proposition 8.7. Let M_0 be a linear subspace of a linear topological space X, and let $x_0 \in X$. Then the function $f : M_0 \to M = x_0 + M_0$ given by $f(m) = x_0 + m$ is a homeomorphism.

Theorem 8.8. In a linear topological space, the closure of a linear subspace is a linear subspace.

Corollary 8.9. In a linear topological space, the closure of a linear variety is a linear variety.

Theorem 8.10. Suppose M is a hyperplane in a linear topological space. If M is not closed, it is dense.

Theorem 8.11. Suppose M is a hyperplane in a linear topological space X, and suppose $f : X \to S$ is a linear function into the scalars such that $M = f^{-1}(t)$ for some scalar t. Then M is closed if and only if f is continuous.