Chapter Nine - Real Linear Spaces and the Hahn- Banach Theorem

Definition. A linear space in which the scalars are the real numbers is called a **real linear space**.

Definition. Suppose $f : X \to \mathbf{R}$ is a linear function from X into the scalars. A set of the form $\{x \in X : f(x) > r\}$ or $\{x \in X : f(x) \geq r\}$ for some $r \in \mathbf{R}$ is called a **half-space.**

Proposition 9.1. The sets $\{x \in X : f(x) < r\}$ and $\{x \in X : f(x) \leq r\}$ are half-spaces.

Definition. Let $M = \{x \in X : f(x) = r\}$ be a hyperplane. Each of the hyperplanes $\{x \in X : f(x) > r\}$, $\{x \in X : f(x) \geq r\}$, $\{x \in X : f(x) < r\}$, $\{x \in X : f(x) > r\}$ is called a **half-space determined by** M.

Definition. A set S is said to **lie on one side** of a hyperplane M if it is included entirely in one of the four half-spaces determined by M. If S lies on one side of M and does not meet M, it is said to **lie strictly on one side of** M**.**

Definition. Two subsets A and B of a real linear space are said to be **separated** by a hyperplane M if each lies on one side of M, but they are not both in the same half-space determined by M. If, in addition, one of the sets lies strictly on one side of M, they are said to be **strictly separated** by M.

Proposition 9.2. A half-space is convex.

Theorem 9.3. If M is a hyperplane in a real linear space X and K is a convex subset of X that does not meet M, then K lies strictly on one side of M.

Definition. If K is a convex subset of a real linear space X, then a point $x_0 \in K$ is called an **internal point** of K if for every $x \in X$, there is an $\varepsilon > 0$ so that $x_0 + sx \in K$ for all s so that $0 \leq s < \varepsilon$.

Proposition 9.4. If K is a convex set, the set K_i of all internal points of K is convex.

Proposition 9.5. Suppose K is a convex subset of a real linear space, x_i is an internal point of K, $y \in K$, and t is a real number with $0 \leq t < 1$. Then $x_0 = (1 - t)x_i + ty$ is an internal point of K.

Proposition 9.6. Suppose A and B are convex subsets of a real linear space X, and suppose A has at least one internal point. If the hyperplane M separates B and the set A_i of internal points of A, then it separates A and B.

Proposition 9.7. Suppose K is a convex subset of a real linear space, and suppose K consists entirely of internal points. If $0 \notin K$, then there is a convex set C having the properties:

i)$K \subset C$;

ii)$0 \notin C$;

iii)C consists entirely of internal points; and

iv)if $\widetilde{C} \subset C$ is another convex set satisfying i), ii), and iii), then $\widetilde{C} = C$.

Proposition 9.8. Suppose C is a convex subset of a real linear space that is maximal with respect to the property $0 \in C$ (That is, if $\widetilde{C}$ is convex, $0 \notin \widetilde{C}$, and $\widetilde{C} \supset C$, then $C = \widetilde{C}$.). If $u \in C$ and $t > 0$, then $tu \in C$.

Proposition 9.9. If, as in Proposition 9.8, C is convex and $0 \notin C$, and C is maximal with respect to these properties, then $M_0 = X \backslash (C \cup (-C))$ is a linear subspace.

Proposition 9.10. Suppose A and B are nonempty disjoint convex subsets of a real linear space X, and suppose A consists entirely of internal points. Then $K = A - B$ is a convex set consisting entirely of internal points, and $0 \notin K$.

Note. $A - B = \{a - b : a \in A \text{ and } b \in B\}$.

Theorem 9.11. Suppose A and B are nonempty convex subsets of a real linear space. If either A or B has an internal point, then there is a hyperplane which separates A and B.

Note. Theorem 9.11 is the celebrated **Hahn-Banach Theorem**.