Math 4640A Final Examination 8/7/99

1. a)Find an upper triangular system equivalent with the system:

$$\begin{bmatrix} 0 & 2\lambda & 2(1+\lambda) & 9 & 7 \\ 1 & 2 & 3 & \lambda & 1 \\ 0 & 2\lambda & 2\lambda & 9 & -3 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2\lambda & 2(1+\lambda) & 9 & 7 \\ 1 & 2 & 3 & \lambda & 1 \\ 0 & 2\lambda & 2\lambda & 9 & -3 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & \lambda & 1 \\ 0 & 2\lambda & 2(1+\lambda) & 9 & 7 \\ 0 & 2\lambda & 2\lambda & 9 & -3 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 3 & \lambda & 1 \\ 0 & 2\lambda & 2(1+\lambda) & 9 & 7 \\ 0 & 2\lambda & 2\lambda & 9 & -3 \\ 0 & -2 & -2 & -\lambda & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & \lambda & 1 \\ 0 & -2 & -2 & -\lambda & -1 \\ 0 & 2\lambda & 2(1+\lambda) & 9 & 7 \\ 0 & 2\lambda & 2\lambda & 9 & -3 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 3 & \lambda & 1 \\ 0 & -2 & -2 & -\lambda & -1 \\ 0 & 0 & 2 & 9-\lambda^2 & 7-\lambda \\ 0 & 0 & 0 & 9-\lambda^2 & -3-\lambda \end{bmatrix}$$

b)Find all λ for which there are solutions, and find all solutions.

First, if $9-\lambda^2 \neq 0$, or $\lambda \neq 3,-3$, the coefficient matrix is invertible and there is exactly one solution for each value of $\lambda \neq 3,-3$:

$x_4 = \frac{-3-\lambda}{9-\lambda^2} = \frac{1}{\lambda-3}$; $x_3 = \frac{1}{2}\left(7-\lambda-\frac{9-\lambda^2}{\lambda-3}\right) = 5$;

$x_2 = -\frac{1}{2}\left(-1+10+\frac{\lambda}{\lambda-3}\right) = -\frac{1}{2}\frac{10\lambda-27}{\lambda-3}$; $x_1 = -5$.

Second, suppose $\lambda = -3$. Then the upper triangular system looks like

$$\begin{bmatrix} 1 & 2 & 3 & -3 & 1 \\ 0 & -2 & -2 & 3 & -1 \\ 0 & 0 & 2 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $0x_4 = 0$, and we see that x_4 can be anything, say $x_4 = t$. Now $x_3 = 5$;

$x_2 = -\frac{1}{2}(-1+10-3t) = -\frac{9}{2}+\frac{3}{2}t$; and

$x_1 = 1 + 3t - 15 + 2\left(-\frac{9}{2} + \frac{3}{2}t\right) = -23 + 6t.$
Finally, for $\lambda = 3$, the last equation becomes $0x_4 = -6$, which clearly has no solutions!

To summarize our results, the system has solutions for every $\lambda \neq 3$, and the solutions are found above.

2. Consider sequences defined by $x_{n+1} = 2 - (1+k)x_n + kx_n^3$.
a)How do you know that if the sequence converges, it converges to 1?

Note simply that 1 is a fixed point of the right-hand side: $g(x) = 2 - (1+k)x + kx^3$.

b)For what values of k can you be sure the sequence will converge for all initial x_0 sufficiently close to 1?

Let's simply look at $|g'(1)|$: $g'(x) = -1 - k + 3kx^2$, and so $g'(1) = -1 + 2k$. We thus want to know for what k will

$$|-1 + 2k| < 1.$$

This means that we want

$$0 < k < 1$$

c)For what k is the convergence of order 2?

In other words, for what k is $g'(1) = 0$ and $g''(1) \neq 0$?

$$g'(1) = -1 + 2k, \text{ and } g''(x) = 6kx.$$

Thus, for $k = \frac{1}{2}$ the order of convergence is 2.
d)For what k is the order of convergence 3?

$g''(1) = 6k$, and $k = 0$ is the only value of k for which this is zero–but the sequence will not converge for this value of k. There are therefore no values of k that give order of convergence 3.

3. If you use Newton's method to solve

$$x^2 - xy + y^2 = 14$$
$$2x^2 + y^2 = 9$$

with an initial guess of $(x_0, y_0) = (1,3)$, what is the next iterate, (x_1, y_1)?

Our system is

$$\mathbf{F}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \end{bmatrix},$$

where
$f_1(x,y) = x^2 - xy + y^2 - 14$ and $f_2(x,y) = 2x^2 + y^2 - 9$. The derivative $\mathbf{F}'(x,y)$ is

$$\mathbf{F}'(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - y & -x + 2y \\ 4x & 2y \end{bmatrix}.$$

Thus,

$$\mathbf{F}'(1,3) = \begin{bmatrix} -1 & 5 \\ 4 & 6 \end{bmatrix}.$$

Now, for Newton's method, we know that $\mathbf{x}_1 = (x_1, y_1) = (x_0, y_0) + (\Delta x, \Delta y)$, where

$$\mathbf{F}'(1,3)\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -\mathbf{F}(1,3).$$

Or,

$$\begin{bmatrix} -1 & 5 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

Solving this is easy:

$$\begin{bmatrix} -1 & 5 \\ 0 & 26 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 7 \\ 26 \end{bmatrix}.$$

Thus,

$$\Delta y = 1, \text{ and } \Delta x = -(7-5) = -2$$

Hence we have it:

$$(x_1, y_1) = (1,3) + (-2,1) = (-1,4).$$

4. a)Find the maximum value of $w(x) = |x(x-h)(x-2h)|$ on the interval $[0, 2h]$.

This is, of course, simply an elementary calculus problem. Note that $w(0) = w(2h) = 0$ so that the maximum value of w on the interval will occur where the derivative of the function $f(x) = x(x-h)(x-2h) = x^3 - 3x^2h + 2xh^2$ is zero. Let's find where this is:

$$f'(x) = 3x^2 - 6xh + 2h^2$$

This is zero at

$$x = \left(1 + \frac{1}{3}\sqrt{3}\right)h, \text{ and } x = \left(1 - \frac{1}{3}\sqrt{3}\right)h$$

Now,

$$\begin{aligned} f\left(\left(1 + \frac{1}{3}\sqrt{3}\right)h\right) &= h^3\left[\left(1 + \frac{1}{3}\sqrt{3}\right)\left(\frac{1}{3}\sqrt{3}\right)\left(-1 + \frac{1}{3}\sqrt{3}\right)\right] \\ &= -\frac{2}{9}\sqrt{3}\,h^3. \end{aligned}$$

Also,

$$f\left(\left(1-\frac{1}{3}\sqrt{3}\right)h\right) = \frac{2}{9}\sqrt{3}\,h^3$$

The maximum value of $w(x)$ is thus

$$\frac{2}{9}\sqrt{3}\,h^3$$

b)We know the error from quadratic interpolation to be given by

$$E = |f(x)-p(x)| = \left|\frac{f^{(3)}(\xi)}{3!}(x-x_0)(x-x_0-h)(x-x_0-2h)\right|,$$

where $0 \le \xi \le \pi/2$. For $f(x) = \sin x$, we have $f^{(3)}(x) = -\cos x$. Thus,

$$\begin{aligned} E &\le \frac{1}{6}|\cos\xi||(x-x_0)(x-x_0-h)(x-x_0-2h)| \\ &\le \frac{1}{6}|(x-x_0)(x-x_0-h)(x-x_0-2h)|. \end{aligned}$$

Let $t = x - x_0$, and we have

$$(x-x_0)(x-x_0-h)(x-x_0-2h) = t(t-h)(t-2h),$$

and so using the result of part a), we have

$$E \le \frac{1}{6}\,\frac{2}{9}\sqrt{3}\,h^3 = \frac{h^3}{9\sqrt{3}}$$

We can be sure $E \le 10^{-6}$ by choosing h so that

$$\frac{h^3}{9\sqrt{3}} < 10^{-6},$$

or,

$$h < 3^{5/6} \bullet 10^{-2}.$$

The number of table entries N should therefore be

$$N \ge \frac{\pi/2}{h} > \frac{\pi/2}{3^{5/6}}100 = \frac{100}{6}\pi\sqrt[6]{3} \approx 62.88$$

Hence we need at least 63 table entries.