Math 2507 - Quiz 7/20 - Solutions

1. Evaluate the integral

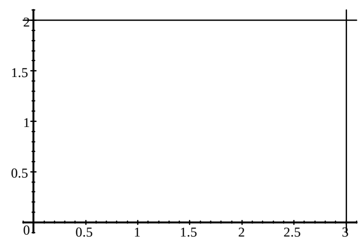
$$\iint_S x^2yzdS,$$

where S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $0 \le x \le 3$, $0 \le y \le 2$.

First, the obvious vector description of S:

$$\mathbf{r}(s,t) = s\mathbf{i} + t\mathbf{j} + (1 + 2s + 3t)\mathbf{k}$$

for (s,t) in the rectangle



Now,

$$\frac{\partial \mathbf{r}}{\partial s} = \mathbf{i} + 2\mathbf{k}, \text{ and } \frac{\partial \mathbf{r}}{\partial t} = \mathbf{j} + 3\mathbf{k}.$$

Thus,

$$\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k},$$

and so $\left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right| = \sqrt{14}$. Our integral now looks like

$$\iint_S x^2yzdS = \int_0^3\int_0^2 s^2t(1 + 2s + 3t)\sqrt{14}\,dtds$$
$$= 171\sqrt{14}$$

2. Find the centroid of the surface S described by $\mathbf{r}(s,t) = s\cos t\mathbf{i} + s\sin t\mathbf{j} + t\mathbf{k}$ for $0 \le s \le 2$, and $0 \le t \le \pi$. (No need to evaluate iterated integrals.)

The centroid $(\tilde{x}, \tilde{y}, \tilde{z})$ is

$$\tilde{x} = \frac{\iint_S xdS}{\iint_S dS},\ \ \tilde{y} = \frac{\iint_S ydS}{\iint_S dS},\ \ \tilde{z} = \frac{\iint_S zdS}{\iint_S dS}.$$

We are given a vector description of the surface so we need only find $\left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right|$:

$$\frac{\partial \mathbf{r}}{\partial s} = \cos t\mathbf{i} + \sin t\mathbf{j}, \text{ and } \frac{\partial \mathbf{r}}{\partial t} = -s\sin t\mathbf{i} + s\cos t\mathbf{j} + \mathbf{k}.$$

Hence,

$$\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 1 \end{vmatrix} = \sin t\mathbf{i} + \cos t\mathbf{j} + s\mathbf{k},$$

and so $\left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right| = \sqrt{1+s^2}$.Thus,

$$\tilde{x} = \frac{\int_0^{\pi}\int_0^{2} s\cos t\sqrt{1+s^2}\, ds dt}{\int_0^{\pi}\int_0^{2} \sqrt{1+s^2}\, ds dt}$$

$$\tilde{y} = \frac{\int_0^{\pi}\int_0^{2} s\sin t\sqrt{1+s^2}\, ds dt}{\int_0^{\pi}\int_0^{2} \sqrt{1+s^2}\, ds dt}$$

$$\tilde{z} = \frac{\int_0^{\pi}\int_0^{2} t\sqrt{1+s^2}\, ds dt}{\int_0^{\pi}\int_0^{2} \sqrt{1+s^2}\, ds dt}$$