Math 2507 Quiz 7/27 Solutions

1. Find the integral $\iint_{\mathbf{S}} (4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}) \bullet d\mathbf{S}$, where **S** is the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$, with the orientation $\mathbf{n}$ such that $\mathbf{n} \bullet \mathbf{k} > 0$.

As usual, we begin by finding a vector description of the surface $\mathbf{S}$:

$$\mathbf{r}(s,t) = s\cos t\mathbf{i} + s\sin t\mathbf{j} + s^2\mathbf{k},$$

where $0 \le s \le 1$, and $0 \le t \le 2\pi$. Now compute $\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}$:

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} &= (\cos t\mathbf{i} + \sin t\mathbf{j} + 2s\mathbf{k}) \times (-s\sin t\mathbf{i} + s\cos t\mathbf{j}) \\ &= -2s^2\cos t\mathbf{i} - 2s^2\sin t\mathbf{j} + s\mathbf{k}\end{aligned}$$

Note that $\left(\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right) \bullet \mathbf{k} = s > 0$, so this is the orientation we seek. Now,

$$\iint_{\mathbf{S}} (4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}) \bullet d\mathbf{S} =$$

$$= \int_0^{2\pi}\int_0^1 (-8s^3 + 2s)\,ds\,dt = -2\pi.$$

2. A plane region is bounded by the simple closed curve **C**. Suppose the line integral

$$\int_{\mathbf{C}} [(3x^2y^2 + 4)\mathbf{i} + (2x^3y + 10x)\mathbf{j}] \bullet d\mathbf{r} = 50,$$

where **C** has the usual counterclockwise orientation. What is the area of the region enclosed by **C**?

Apply Green's Theorem to the line integral:

$$\int_{\mathbf{C}} [(3x^2y^2 + 4)\mathbf{i} + (2x^3y + 10x)\mathbf{j}] \bullet d\mathbf{r} =$$

$$\begin{aligned}&= \iint_R [6x^2y + 10 - 6x^2y]\,dA \\ &= \iint_R 10\,dA = 10\iint_R dA = 50.\end{aligned}$$

Hence,

$$Area = \iint_R dA = \frac{50}{10} = 5.$$