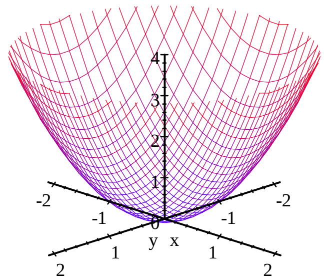
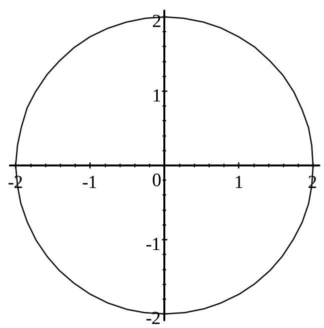
Math 2507A Quiz Two Solutions

1. Let V be the volume of the solid bounded below by $z = x^2 + y^2$ and bounded above by $z = 4$.

a)Give an iterated integral for V in which the first, or "inside" integration is with respect to z.



When we project onto the $x-y$ plane, we see $4 = x^2 + y^2$, a circle of radius 2, centered at the origin:



Then $V = \iint\limits_C \left(\int_{x^2+y^2}^{4} dz \right) dA$, where C is the region enclosed by the circle $x^2 + y^2 = 4$.

There are several correct ways to proceed.

In rectangular coordinates, we have

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} dzdydx, \text{ or}$$

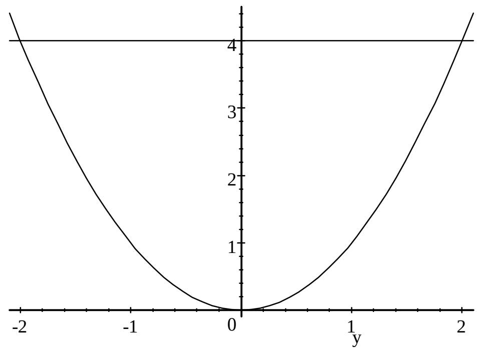
$$V = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{x^2+y^2}^{4} dzdxdy, \text{ or}$$

In polar coordinates:

$$V = \int_0^{2\pi}\int_0^2\int_{r^2}^4 r\,dz\,dr\,d\theta$$

a)Give an iterated integral for V in which the first, or "inside" integration is with respect to x.

Now we project the solid onto the $y-z$ plane:



Now, $V = \iint_R \left(\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx \right)$, where R is the region in the picture. Thus we have either

$$V = \int_{-2}^{2}\int_{y^2}^{4}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dz\,dy, \text{ or}$$

$$V = \int_{0}^{4}\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dy\,dz, \text{ or}$$

2. A wire has the shape of the curve $y = x^3$, for $0 \le x \le 2$. The density of the wire is given by $\rho(x,y) = 3x^3 + y$. Find the mass of the wire.

The mass M is simply $\int_W \rho(x,y)dr$. So,

$$M = \int_W \rho(x,y)dr = \int_a^b \rho(\mathbf{r}(t))|\mathbf{r}'(t)|dt,$$

where $\mathbf{r}(t)$, $a \le t \le b$, is a vector description of the curve $y = x^3$. For a vector description, simply use $t = x$:

$$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j},\ 0 \le t \le 2.$$

Now, $\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$, and so $|\mathbf{r}'(t)| = \sqrt{1+9t^4}$.

Next, we have $\rho(\mathbf{r}(t)) = 3t^3 + t^3 = 4t^3$, and so our integral becomes

$$M = \int_a^b \rho(\mathbf{r}(t))|\mathbf{r}'(t)|dt = \int_0^2 4t^3\sqrt{1+9t^4}\,dt$$

$$= \frac{2}{27}(1+9t^4)^{3/2}\Big|_0^2 = \frac{2}{27}[(145)^{3/2} - 1]$$

$$= \frac{290}{27}\sqrt{145} - \frac{2}{27}$$

Finis