Chapter Eight

Series

8.1. Sequences. The basic definitions for complex sequences and series are essentially the same as for the real case. A **sequence** of complex numbers is a function $g : Z_+ \to \mathbf{C}$ from the positive integers into the complex numbers. It is traditional to use subscripts to indicate the values of the function. Thus we write $g(n) \equiv z_n$ and an explicit name for the sequence is seldom used; we write simply (z_n) to stand for the sequence g which is such that $g(n) = z_n$. For example, $(\frac{i}{n})$ is the sequence g for which $g(n) = \frac{i}{n}$.

The number L is a **limit** of the sequence (z_n) if given an $\varepsilon > 0$, there is an integer N_ε such that $|z_n - L| < \varepsilon$ for all $n \geq N_\varepsilon$. If L is a limit of (z_n), we sometimes say that (z_n) **converges** to L. We frequently write $\lim(z_n) = L$. It is relatively easy to see that if the complex sequence $(z_n) = (u_n + iv_n)$ converges to L, then the two real sequences (u_n) and (v_n) each have a limit: (u_n) converges to $\operatorname{Re}L$ and (v_n) converges to $\operatorname{Im}L$. Conversely, if the two real sequences (u_n) and (v_n) each have a limit, then so also does the complex sequence $(u_n + iv_n)$. All the usual nice properties of limits of sequences are thus true:

$$\begin{aligned} \lim(z_n \pm w_n) &= \lim(z_n) \pm \lim(w_n); \\ \lim(z_n w_n) &= \lim(z_n)\lim(w_n); \text{ and} \\ \lim\left(\frac{z_n}{w_n}\right) &= \frac{\lim(z_n)}{\lim(w_n)}. \end{aligned}$$

provided that $\lim(z_n)$ and $\lim(w_n)$ exist. (And in the last equation, we must, of course, insist that $\lim(w_n) \neq 0$.)

A necessary and sufficient condition for the convergence of a sequence (a_n) is the celebrated **Cauchy criterion**: given $\varepsilon > 0$, there is an integer N_ε so that $|a_n - a_m| < \varepsilon$ whenever $n, m > N_\varepsilon$.

A sequence (f_n) of functions on a domain D is the obvious thing: a function from the positive integers into the set of complex functions on D. Thus, for each $z \epsilon D$, we have an ordinary sequence $(f_n(z))$. If each of the sequences $(f_n(z))$ converges, then we say the sequence of functions (f_n) converges to the function f defined by $f(z) = \lim(f_n(z))$. This pretty obvious stuff. The sequence (f_n) is said to converge to f **uniformly** on a set S if given an $\varepsilon > 0$, there is an integer N_ε so that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N_\varepsilon$ and all $z \in S$.

Note that it is possible for a sequence of continuous functions to have a limit function that is *not* continuous. This cannot happen if the convergence is uniform. To see this, suppose the sequence (f_n) of continuous functions converges uniformly to f on a domain D, let $z_0 \epsilon D$, and let $\varepsilon > 0$. We need to show there is a δ so that $|f(z_0) - f(z)| < \varepsilon$ whenever

$|z_0 - z| < \delta$. Let's do it. First, choose N so that $|f_N(z) - f(z)| < \frac{\varepsilon}{3}$. We can do this because of the uniform convergence of the sequence (f_n). Next, choose δ so that $|f_N(z_0) - f_N(z)| < \frac{\varepsilon}{3}$ whenever $|z_0 - z| < \delta$. This is possible because f_N is continuous. Now then, when $|z_0 - z| < \delta$, we have

$$\begin{aligned} |f(z_0) - f(z)| &= |f(z_0) - f_N(z_0) + f_N(z_0) - f_N(z) + f_N(z) - f(z)| \\ &\leq |f(z_0) - f_N(z_0)| + |f_N(z_0) - f_N(z)| + |f_N(z) - f(z)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

and we have done it!

Now suppose we have a sequence (f_n) of continuous functions which converges uniformly on a contour C to the function f. Then the sequence $\left(\int_C f_n(z)dz\right)$ converges to $\int_C f(z)dz$. This is easy to see. Let $\varepsilon > 0$. Now let N be so that $|f_n(z) - f(z)| < \frac{\varepsilon}{A}$ for $n > N$, where A is the length of C. Then,

$$\begin{aligned} \left|\int_C f_n(z)dz - \int_C f(z)dz\right| &= \left|\int_C (f_n(z) - f(z))dz\right| \\ &< \frac{\varepsilon}{A}A = \varepsilon \end{aligned}$$

whenever $n > N$.

Now suppose (f_n) is a sequence of functions each *analytic* on some region D, and suppose the sequence converges uniformly on D to the function f. Then f is analytic. This result is in marked contrast to what happens with real functions—examples of uniformly convergent sequences of differentiable functions with a nondifferentiable limit abound in the real case. To see that this uniform limit is analytic, let $z_0 \epsilon D$, and let $S = \{z : |z - z_0| < r\} \subset D$. Now consider any simple closed curve $C \subset S$. Each f_n is analytic, and so $\int_C f_n(z)dz = 0$ for every n. From the uniform convergence of (f_n), we know that $\int_C f(z)dz$ is the limit of the sequence $\left(\int_C f_n(z)dz\right)$, and so $\int_C f(z)dz = 0$. Morera's theorem now tells us that f is analytic on S, and hence at z_0. Truly a miracle.

Exercises

1. Prove that a sequence cannot have more than one limit. (We thus speak of *the* limit of a sequence.)

2. Give an example of a sequence that does not have a limit, or explain carefully why there is no such sequence.

3. Give an example of a bounded sequence that does not have a limit, or explain carefully why there is no such sequence.

4. Give a sequence (f_n) of functions continuous on a set D with a limit that is not continuous.

5. Give a sequence of real functions differentiable on an interval which converges uniformly to a nondifferentiable function.

8.2 Series. A series is simply a sequence (s_n) in which $s_n = a_1 + a_2 + \ldots + a_n$. In other words, there is sequence (a_n) so that $s_n = s_{n-1} + a_n$. The s_n are usually called the **partial sums.** Recall from Mrs. Turner's class that if the series $\left(\sum_{j=1}^{n} a_j\right)$ has a limit, then it must be true that $\lim_{n\to\infty} (a_n) = 0$.

Consider a series $\left(\sum_{j=1}^{n} f_j(z)\right)$ of functions. Chances are this series will converge for some values of z and not converge for others. A useful result is the celebrated **Weierstrass M-test**: Suppose (M_j) is a sequence of real numbers such that $M_j \geq 0$ for all $j > J$, where J is some number., and suppose also that the series $\left(\sum_{j=1}^{n} M_j\right)$ converges. If for all $z \epsilon D$, we have $|f_j(z)| \leq M_j$ for all $j > J$, then the series $\left(\sum_{j=1}^{n} f_j(z)\right)$ converges uniformly on D.

To prove this, begin by letting $\varepsilon > 0$ and choosing $N > J$ so that

$$\sum_{j=m}^{n} M_j < \varepsilon$$

for all $n, m > N$. (We can do this because of the famous Cauchy criterion.) Next, observe that

$$\left|\sum_{j=m}^{n} f_j(z)\right| \leq \sum_{j=m}^{n} |f_j(z)| \leq \sum_{j=m}^{n} M_j < \varepsilon.$$

This shows that $\left(\sum_{j=1}^{n} f_j(z)\right)$ converges. To see the uniform convergence, observe that

$$\left|\sum_{j=m}^{n} f_j(z)\right| = \left|\sum_{j=0}^{n} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| < \varepsilon$$

for all $z \epsilon D$ and $n > m > N$. Thus,

$$\lim_{n\to\infty} \left|\sum_{j=0}^{n} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| = \left|\sum_{j=0}^{\infty} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| \leq \varepsilon$$

for $m > N$.(The limit of a series $\left(\sum_{j=0}^{n} a_j\right)$ is almost always written as $\sum_{j=0}^{\infty} a_j$.)

Exercises

6. Find the set D of all z for which the sequence $\left(\frac{z^n}{z^n - 3^n}\right)$ has a limit. Find the limit.

7. Prove that the series $\left(\sum_{j=1}^{n} a_j\right)$ converges if and only if both the series $\left(\sum_{j=1}^{n} \operatorname{Re} a_j\right)$ and $\left(\sum_{j=1}^{n} \operatorname{Im} a_j\right)$ converge.

8. Explain how you know that the series $\left(\sum_{j=1}^{n} \left(\frac{1}{z}\right)^j\right)$ converges uniformly on the set $|z| \geq 5$.

8.3 Power series. We are particularly interested in series of functions in which the partial sums are polynomials of increasing degree:

$$s_n(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots + c_n(z - z_0)^n.$$

(We start with $n = 0$ for esthetic reasons.) These are the so-called **power series**. Thus, a power series is a series of functions of the form $\left(\sum_{j=0}^{n} c_j(z-z_0)^j\right)$.

Let's look first at a very special power series, the so-called **Geometric series:**

$$\left(\sum_{j=0}^{n} z^j\right).$$

Here

$$\begin{aligned} s_n &= 1 + z + z^2 + \ldots + z^n, \text{ and} \\ zs_n &= z + z^2 + z^3 + \ldots + z^{n+1}. \end{aligned}$$

Subtracting the second of these from the first gives us

$$(1-z)s_n = 1 - z^{n+1}.$$

If $z = 1$, then we can't go any further with this, but I hope it's clear that the series does not have a limit in case $z = 1$. Suppose now $z \neq 1$. Then we have

$$s_n = \frac{1}{1-z} - \frac{z^{n+1}}{1-z}.$$

Now if $|z| < 1$, it should be clear that $\lim(z^{n+1}) = 0$, and so

$$\lim\left(\sum_{j=0}^{n} z^j\right) = \lim s_n = \frac{1}{1-z}.$$

Or,

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \text{ for } |z| < 1.$$

There is a bit more to the story. First, note that if $|z| > 1$, then the Geometric series does not have a limit (why?). Next, note that if $|z| \leq \rho < 1$, then the Geometric series converges *uniformly* to $\frac{1}{1-z}$. To see this, note that

$$\left(\sum_{j=0}^{n} \rho^j \right)$$

has a limit and appeal to the Weierstrass M-test.

Clearly a power series will have a limit for some values of z and perhaps not for others. First, note that any power series has a limit when $z = z_0$. Let's see what else we can say. Consider a power series $\left(\sum_{j=0}^{n} c_j (z - z_0)^j \right)$. Let

$$\lambda = \limsup \left(\sqrt[j]{|c_j|} \right).$$

(Recall from 6th grade that $\limsup(a_k) = \lim(\sup\{a_k : k \geq n\}.)$ Now let $R = \frac{1}{\lambda}$. (We shall say $R = 0$ if $\lambda = \infty$, and $R = \infty$ if $\lambda = 0$.) We are going to show that the series converges uniformly for all $|z - z_0| \leq \rho < R$ and diverges for all $|z - z_0| > R$.

First, let's show the series does not converge for $|z - z_0| > R$. To begin, let k be so that

$$\frac{1}{|z - z_0|} < k < \frac{1}{R} = \lambda.$$

There are an infinite number of c_j for which $\sqrt[j]{|c_j|} > k$, otherwise $\limsup \left(\sqrt[j]{|c_j|} \right) \leq k$. For each of these c_j we have

$$|c_j (z - z_0)^j| = \left(\sqrt[j]{|c_j|}\, |z - z_0| \right)^j > (k|z - z_0|)^j > 1.$$

It is thus not possible for $\lim_{n \to \infty} |c_n (z - z_0)^n| = 0$, and so the series does not converge.

Next, we show that the series *does* converge uniformly for $|z - z_0| \leq \rho < R$. Let k be so that

$$\lambda = \frac{1}{R} < k < \frac{1}{\rho}.$$

Now, for j large enough, we have $\sqrt[j]{|c_j|} < k$. Thus for $|z - z_0| \leq \rho$, we have

$$|c_j(z-z_0)^j| = \left(\sqrt[j]{|c_j|}\,|z-z_0|\right)^j < (k|z-z_0|)^j < (k\rho)^j.$$

The geometric series $\left(\sum_{j=0}^{n}(k\rho)^j\right)$ converges because $k\rho < 1$ and the uniform convergence of $\left(\sum_{j=0}^{n}c_j(z-z_0)^j\right)$ follows from the M-test.

Example

Consider the series $\left(\sum_{j=0}^{n}\frac{1}{j!}z^j\right)$. Let's compute $R = 1/\limsup\left(\sqrt[j]{|c_j|}\right) = \limsup(\sqrt[j]{j!})$. Let K be any positive integer and choose an integer m large enough to insure that $2^m > \frac{K^{2K}}{(2K)!}$. Now consider $\frac{n!}{K^n}$, where $n = 2K+m$:

$$\begin{aligned}\frac{n!}{K^n} &= \frac{(2K+m)!}{K^{2K+m}} = \frac{(2K+m)(2K+m-1)\dots(2K+1)(2K)!}{K^mK^{2K}}\\ &> 2^m\frac{(2K)!}{K^{2K}} > 1\end{aligned}$$

Thus $\sqrt[n]{n!} > K$. Reflect on what we have just shown: given any number K, there is a number n such that $\sqrt[n]{n!}$ is bigger than it. In other words, $R = \limsup(\sqrt[j]{j!}) = \infty$, and so the series $\left(\sum_{j=0}^{n}\frac{1}{j!}z^j\right)$ converges for all z.

Let's summarize what we have. For any power series $\left(\sum_{j=0}^{n}c_j(z-z_0)^j\right)$, there is a number $R = \frac{1}{\limsup\left(\sqrt[j]{|c_j|}\right)}$ such that the series converges uniformly for $|z-z_0| \le \rho < R$ and does not converge for $|z-z_0| > R$. (Note that we may have $R = 0$ or $R = \infty$.) The number R is called the **radius of convergence** of the series, and the set $|z-z_0| = R$ is called the **circle of convergence.** Observe also that the limit of a power series is a function analytic inside the circle of convergence (why?).

Exercises

9. Suppose the sequence of real numbers (α_j) has a limit. Prove that

$$\limsup(\alpha_j) = \lim(\alpha_j).$$

For each of the following, find the set D of points at which the series converges:

10. $\left(\sum_{j=0}^{n} j!z^j\right)$.

11. $\left(\sum_{j=0}^{n} jz^j\right)$.

12. $\left(\sum_{j=0}^{n} \frac{j^2}{3^j} z^j\right)$.

13. $\left(\sum_{j=0}^{n} \frac{(-1)^j}{2^{2j}(j!)^2} z^{2j}\right)$

8.4 Integration of power series. Inside the circle of convergence, the limit

$$S(z) = \sum_{j=0}^{\infty} c_j(z - z_0)^j$$

is an analytic function. We shall show that this series may be integrated ”term-by-term”—that is, the integral of the limit is the limit of the integrals. Specifically, if C is any contour inside the circle of convergence, and the function g is continuous on C, then

$$\int_C g(z)S(z)dz = \sum_{j=0}^{\infty} c_j \int_C g(z)(z - z_0)^j dz.$$

Let’s see why this. First, let $\varepsilon > 0$. Let M be the maximum of $|g(z)|$ on C and let L be the length of C. Then there is an integer N so that

$$\left|\sum_{j=n}^{\infty} c_j(z - z_0)^j\right| < \frac{\varepsilon}{ML}$$

for all $n > N$. Thus,

$$\left| \int_C \left(g(z) \sum_{j=n}^{\infty} c_j (z - z_0)^j \right) dz \right| < ML \frac{\varepsilon}{ML} = \varepsilon,$$

Hence,

$$\left| \int_C g(z) S(z) dz - \sum_{j=0}^{n-1} c_j \int_C g(z)(z - z_0)^j dz \right| = \left| \int_C \left(g(z) \sum_{j=n}^{\infty} c_j (z - z_0)^j \right) dz \right|$$
$$< \varepsilon,$$

and we have shown what we promised.

8.5 Differentiation of power series. Again, let

$$S(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j.$$

Now we are ready to show that inside the circle of convergence,

$$S'(z) = \sum_{j=1}^{\infty} j c_j (z - z_0)^{j-1}.$$

Let z be a point inside the circle of convergence and let C be a positive oriented circle centered at z and inside the circle of convergence. Define

$$g(s) = \frac{1}{2\pi i (s - z)^2},$$

and apply the result of the previous section to conclude that

$$\int_C g(s)S(s)ds = \sum_{j=0}^{\infty} c_j \int_C g(s)(s-z_0)^j ds, \text{or}$$

$$\frac{1}{2\pi i}\int_C \frac{S(s)}{(s-z)^2}ds = \sum_{j=0}^{\infty} c_j \frac{1}{2\pi i}\int_C \frac{(s-z_0)^j}{(s-z)^2}ds. \text{ Thus}$$

$$S'(z) = \sum_{j=0}^{\infty} jc_j(z-z_0)^{j-1},$$

as promised!

Exercises

14. Find the limit of

$$\left(\sum_{j=0}^{n}(j+1)z^j\right).$$

For what values of z does the series converge?

15. Find the limit of

$$\left(\sum_{j=1}^{n}\frac{z^j}{j}\right).$$

For what values of z does the series converge?

16. Find a power series $\left(\sum_{j=0}^{n} c_j(z-1)^j\right)$ such that

$$\frac{1}{z} = \sum_{j=0}^{\infty} c_j(z-1)^j, \text{ for } |z-1| < 1.$$

17. Find a power series $\left(\sum_{j=0}^{n} c_j(z-1)^j\right)$ such that

$$\text{Log}\, z = \sum_{j=0}^{\infty} c_j (z-1)^j, \text{ for } |z-1| < 1.$$