Math 4320A **Final Examination** **Winter 1999**

You may use any books, notes, tables, or calculators you wish. Give *exact* answers—do not give decimal or other approximations. *Fortuna vobiscum*.

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1. Find all values of i^i.

$i^i = e^{i\log i} = e^{i(\ln|i|+\arg i)} = e^{i(\pi/2+2k\pi)i} = e^{-(1+4k)\pi/2}$, for $k = 0, \pm 1, \pm 2, \ldots$.

2. Evaluate $\int_C (z+2\bar{z})dz$, where C is the path from $z = 0$ to $z = 1+2i$ consisting of the line segment from 0 to 1 together with the segment from 1 to $1+2i$.

$\int_C (z+2\bar{z})dz = \int_{L_1} (z+2\bar{z})dz + \int_{L_2} (z+2\bar{z})dz$, where L_1 is the segment from 0 to 1, and L_2 is the segment from 1 to $1+2i$.

A complex description of L_1 is simply $\gamma_1(t) = t$, for $0 \le t \le 1$. Thus,

$\int_{L_1} (z+2\bar{z})dz = \int_0^1 (\gamma_1(t) + 2\overline{\gamma_1(t)})\gamma_1'(t)dt = \int_0^1 (t+2t)dt = \frac{3}{2}t^2 \Big|_0^1 = \frac{3}{2}$.

A complex description of L_2 is $\gamma_2(t) = 1+2ti$, for $0 \le t \le 1$. Thus,

$$\int_{L_2} (z+2\bar{z})dz = \int_0^1 (\gamma_2(t) + 2\overline{\gamma_2(t)})\gamma_2'(t)dt = \int_0^1 (1+2ti+2-4ti)2idt$$

$$= \int_0^1 (6i+4t)dt = 6i+2.$$

Hence,

$$\int_C (z+2\bar{z})dz = \int_{L_1} (z+2\bar{z})dz + \int_{L_2} (z+2\bar{z})dz = \frac{7}{2} + 6i.$$

3. Show that the function f defined by

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

is analytic at $z = 0$, and find the derivative $f'(0)$.

For all z, we have $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$. Thus,

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots,$$

and we see that

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$$

is the limit of a power series valid for all z. Hence f is, in fact, entire, and we see that $f'(0) = 0$.

4. Let

$$f(z) = \frac{1}{z^2(z+2i)}.$$

a)Find a Laurent series in powers of z which converges to f and specify the region on which the series converges.

$$\frac{1}{z^2(z+2i)} = \frac{1}{z^2} \bullet \frac{1}{(z+2i)}$$

.

Now,

$$\frac{1}{(z+2i)} = \frac{1}{2i}\left[\frac{1}{1+\frac{z}{2i}}\right] = \frac{1}{2i}\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{(2i)^k}\text{ , for }\left|\frac{z}{2i}\right| < 1,\text{or }|z| < 2.$$

We now have the Laurent series

$$f(z) = \frac{1}{z^2}\frac{1}{2i}\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{(2i)^k} = \sum_{k=0}^{\infty}\frac{(-1)^k z^{k-2}}{(2i)^{k+1}} = \sum_{k=-2}^{\infty}\frac{(-1)^{k+2}z^k}{(2i)^{k+3}}\text{, valid for } 0 < |z| < 2.$$

b)Find another Laurent series in powers of z which converges to f and specify the region on which the series converges.

Here consider,

$$\frac{1}{(z+2i)} = \frac{1}{z}\left[\frac{1}{1+\frac{2i}{z}}\right] = \frac{1}{z}\sum_{k=0}^{\infty}\frac{(-1)^k(2i)^k}{z^k} = \sum_{k=0}^{\infty}\frac{(-1)^k(2i)^k}{z^{k+1}} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2i)^{k-1}}{z^k}.$$

This is valid for $\left|\frac{2i}{z}\right| < 1$, or $2 < |z|$. Then,

$$f(x) = \frac{1}{z^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2i)^{k-1}}{z^k} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2i)^{k-1}}{z^{k+2}} = \sum_{k=3}^{\infty}\frac{(-2i)^{k-3}}{z^k}\text{ , for } 2 < |z|.$$

5. Suppose C is the circle $|z| = 5$ with the usual positive orientation. Evaluate the integrals:

a)

$$\int_C \sin\left(\frac{1}{z}\right)dz.$$

There is precisely one singular point inside C and so the value of the integral is $2\pi i$ times the residue at this point, $z = 0$. The Laurent expansion here is easy—it si simply

$$\sin(\tfrac{1}{z}) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \dots$$

The residue is thus 1, and so

$$\int_C \sin\left(\frac{1}{z}\right) dz = 2\pi i$$

b)

$$\int_C z \sin\left(\frac{1}{z}\right) dz.$$

The Laurent series at 0 is

$$z \sin(\tfrac{1}{z}) = 1 - \frac{1}{3!z^2} + \frac{1}{5!z^4} - \dots,$$

and so the residue at 0 is 0. Hence,

$$\int_C z \sin\left(\frac{1}{z}\right) dz = 0.$$

c)

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz.$$

Here we have

$$z^2 \sin(\tfrac{1}{z}) = z - \frac{1}{3!z} + \frac{1}{5!z^3} - \dots,$$

and we see the residue is $\frac{-1}{3!} = -\frac{1}{6}$. Hence,

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz = 2\pi i\left(\frac{-1}{6}\right) = -\frac{\pi i}{3}.$$