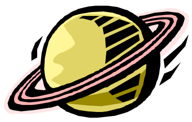
Math 4320A Quiz One January 29, 1999

You may use any books, notes, tables, calculators, or whatever, you wish. Please write so that someone other than yourself can understand your exposition. Give exact answers—do not give decimal or other approximations. If you give approximations, I shall make every effort to have you expelled from Georgia Tech.

Please do not return this sheet.



1. Let

$$f(z) = \begin{cases} \frac{z}{|z|} & \text{for } \ z \neq 0 \\ 0 & \text{for } \ z = 0 \end{cases}$$

a)At what points do the real and imaginary parts of f satisfy the Cauchy-Riemann Equations?

$$f(z) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}} & \text{for } \ (x,y) \neq 0 \\ 0 & \text{for } \ (x,y) = 0 \end{cases}$$

Thus,

$$u(x,y) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}} & \text{for } \ (x,y) \neq 0 \\ 0 & \text{for } \ (x,y) = 0 \end{cases}, \text{ and}$$

$$v(x,y) = \begin{cases} \frac{y}{\sqrt{x^2+y^2}} & \text{for } \ (x,y) \neq 0 \\ 0 & \text{for } \ (x,y) = 0 \end{cases}.$$

First, look at the points other than $(x,y) = 0$. Then $u(x,y) = \frac{x}{\sqrt{x^2+y^2}}$, so $\frac{\partial u(x,y)}{\partial x} = \frac{y^2}{\left(\sqrt{(x^2+y^2)}\right)^3}$ and $\frac{\partial u(x,y)}{\partial y} = -\frac{xy}{\left(\sqrt{(x^2+y^2)}\right)^3}$. Also, $\frac{\partial v(x,y)}{\partial x} = -\frac{xy}{\left(\sqrt{(x^2+y^2)}\right)^3}$ and $\frac{\partial v(x,y)}{\partial y} = \frac{x^2}{\left(\sqrt{(x^2+y^2)}\right)^3}$. The Cauchy-Riemann Equations now look like

$$\frac{y^2}{\left(\sqrt{(x^2+y^2)}\right)^3} = \frac{x^2}{\left(\sqrt{(x^2+y^2)}\right)^3},$$
$$-\frac{xy}{\left(\sqrt{(x^2+y^2)}\right)^3} = \frac{xy}{\left(\sqrt{(x^2+y^2)}\right)^3}.$$

Or,

$$y^2 = x^2$$
$$-xy = xy$$

The second of these equations becomes $2xy = 0$, which means either x or y must $= 0$. But then from the first equation, they are both 0—there are thus no solutions for $(x, y) \neq 0$. Thus the Cauchy-Riemann equations are satisfied for no points, except possibly $z = 0$. Let's take a look at what happens there:

$$\begin{aligned}\frac{\partial u(0,0)}{\partial x} &= \lim_{\Delta x \to 0} \frac{u(0+\Delta x, 0) - u(0,0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{\Delta x}{|\Delta x|}}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{|\Delta x|}\end{aligned}$$

which clearly does not exist. Hence the Cauchy-Riemann equations cannot possibly be satisfied at $z = 0$. Thus, there are *no points* at which the C-R equations are satisfied.

b)At what points is f differentiable?

If f is differentiable at z, then the real and imaginary parts of f must satisfy the Cauchy-Riemann equations—we have just shown there are no such points. Thus there are *no points* at which f is differentiable.

c)At what points is f analytic?

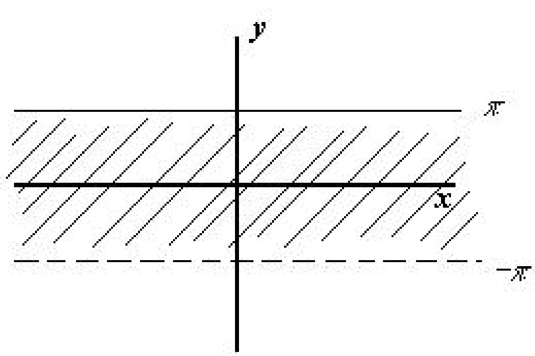
None!

2. Sketch the set of all z for which $\mathrm{Log}(e^z) = z$.

Now, $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$. Thus, $|e^z| = e^x$, and $\mathrm{Arg}(e^z) = y + 2n\pi$ for some integer n. We must have

$$Log(e^z) = \ln|e^x| + i\mathrm{Arg}(e^z) = x + \mathrm{Arg}(e^z) = x + iy$$

This means we need to have $y + 2n\pi = y$, so $n = 0$. Then $y = \mathrm{Arg}(e^z)$, which means $-\pi < y \leq \pi$. Now for the picture:



3. a)Find all values of 1^5.

$1^5 = e^{5\log(1)} = e^{5(\ln 1 + i \arg 1)} = e^{5i \arg 1} = e^{i10n\pi} = \cos 10n\pi + i \sin 10n\pi = 1.$

b)Find all values of $1^{3/4}$.

$1^{3/4} = e^{(3/4)\log 1} = e^{(3/4)i2n\pi}$, for, of course, $n = 0, \pm 1, \pm 2, \ldots$. Then,
$e^{(3/4)i2n\pi} = e^{i(3/2)n\pi} = \cos\left(\frac{3}{2}n\pi\right) + i\sin\left(\frac{3}{2}n\pi\right)$

For $n = 0$, this is 1.

For $n = 1$, this is $\cos\left(\frac{3}{2}\pi\right) + i\sin\left(\frac{3}{2}\pi\right) = -i$.

For $n = 2$, this is $\cos(3\pi) + i\sin(3\pi) = -1$.

For $n = 3$, this is $\cos\left(\frac{9}{2}\pi\right) + i\sin\left(\frac{9}{2}\pi\right) = i$.

For $n = 4$, this $\cos(6\pi) + i\sin(6\pi) = \cos(0) + i\sin(0) = 1$, the same as we got for $n = 0$. Similarly, for $n = 5$, we have $\cos\left(\frac{15}{2}\pi\right) + i\sin\left(\frac{15}{2}\pi\right) = \cos\left(\frac{3}{2}\pi\right) + i\sin\left(\frac{3}{2}\pi\right)$, and we get nothing new from now on. Similarly, for negative integers we get nothing new. There are thus four values for $1^{3/4}$ and they are $1, -i, -1,$ and i.

c)Find all values of $1^{\sqrt{2}}$.

$1^{\sqrt{2}} = e^{\sqrt{2}\log 1} = e^{\sqrt{2}i2n\pi} = \cos(2\sqrt{2}n\pi) + i\sin(2\sqrt{2}n\pi)$, for $n = 0, \pm 1, \pm 2, \ldots$.

(There are an infinite number of these!)

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