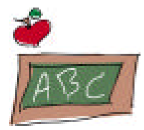
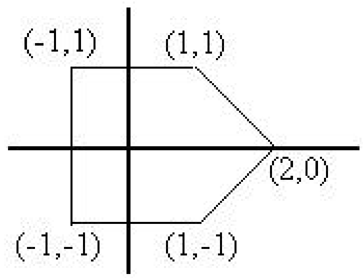
Math 4320A Quiz Two February 26, 1999

You may use any books, notes, tables, calculators, or whatever, you wish. Please write so that someone other than yourself can understand your exposition. Give exact answers—do not give decimal or other approximations. If you give approximations, I shall make every effort to have you expelled from Georgia Tech.

Please do not return this sheet.



1. Let C be the contour pictured:



(with the counterclockwise orientation). Evaluate the integral

$$\int_C \frac{1}{z^2+2z}\,dz.$$

Simply appeal to the Cauchy Integral formula: the function $f(z) = \frac{1}{z+2}$ is analytic on and inside the contour C, and so

$$\int_C \frac{1}{z^2+2z}\,dz = \int_C \frac{f(z)}{z-0}\,dz = 2\pi i f(0) = 2\pi i \frac{1}{2} = \pi i$$

2. Find an entire function f such that $\operatorname{Re} f(z) = x^2 + 2x - y^2 + 1$, or explain carefully why there is no such function.

It is easy to verify that $u(x,y) = x^2 + 2x - y^2 + 1$ is harmonic everywhere, and so there must exist an entire $f = u + iv$. Let's find it. From the Cauchy-Riemann equations, we know that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y.$$

Hence,

$$v(x,y) = 2xy + h(y).$$

Now,

$$\frac{\partial v}{\partial y} = 2x + h'(y) = \frac{\partial u}{\partial x} = 2x + 2.$$

Thus,

$$h'(y) = 2, \text{ and so } h(y) = 2y.$$

Finally,

$$v(x,y) = 2xy + h(y) = 2xy + 2y, \text{ or}$$
$$f(z) = u + iv = x^2 + 2x - y^2 + 1 + i(2xy + 2y)$$

3. Suppose f is analytic on and inside a simple closed curve C. Suppose moreover that $|f(z) - 1| < 1$ for all $z \epsilon C$. Explain why $f(z) \neq 0$ for all z inside C.

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Let $g(z) = f(z) - 1$. Then $|g(s)| < 1$ for all $s \epsilon C$, and so from the Maximum Modulus Theorem we know that $|g(z)| < 1$ for all z *inside* C. If there were a z_0 inside C for which $f(z_0) = 0$, this would mean that $|g(z_0)| = |f(z_0) - 1| = |0 - 1| = 1$, a contradiction. Thus there is no such z_0.

4. Does the function f defined by $f(z) = \frac{1}{z}$ have an antiderivative on the set of all $z \neq 0$? Explain. Does f have an antiderivative on the set of all z for which $\operatorname{Re} z < 0$? Explain.

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If f has an antiderivative in some region D, then we know that $\int_C f(z)dz = 0$ for every closed curve C included in D. But we know that C, the circle of radius 1 and centered at the origin is included in the set of all $z \neq 0$, and we know, or can easily show again, that $\int_C \frac{1}{z} dz = 2\pi i \neq 0$. In other words, f does *not* have an antiderivative on this set.

On the other hand, define $F(z) = \log z$, for $0 < \arg z < 2\pi$. Then F is analytic in the set of all z for which $\operatorname{Re} z < 0$, and $F'(z) = \frac{1}{z} = f(z)$ there.

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